# Localised variance in target number for the Cardinalized Probability Hypothesis Density filter

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***Abstract*—Following a recent study on the Probability Hypothesis Density filter, this paper aims at extracting higher-order information statistics on the local target number from the filtered state of the Cardinalized Probability Hypothesis Density filter, based on recent developments of novel derivation tools in the multi-object filtering framework. In addition to the description of a novel approach for retrieving the expression of the updated localised mean target number, this paper proposes the extraction of the novel *localised variance in the target number* across the whole state space.**

***Keywords*—Multi-object filtering, Higher-order statistics, CPHD filter**

## I. Introduction

The exploitation of the Random Finite Set (RFS) theory for multi-target detection and tracking problems was popularized by Mahler's Probability Hypothesis Density (PHD) filter [1]. The tracking community has enjoyed ever since a vast and resourceful framework, rigorously described by the Finite Set Statistics (FISST), for the design and implementation of multi-object filters in challenging environments, notably when the number of targets is unknown and possibly varying.

The PHD filter propagates the *probability hypothesis density* (or *intensity*) of a *point process* describing the multi-target configuration in the state space, and thus provides an estimation of the mean target number in any region of the state space. Elegant and computationally inexpensive, it has been enjoying a wide popularity for the design of tracking applications in various domains. More recently, the relative instability of the PHD filter in the estimation of the target number led Mahler to the design of the Cardinalized Probability Hypothesis Density (CPHD) filter [2]. In addition to the intensity, it propagates the full cardinality distribution of the global target number, assuming that the filtered process can be approximated by an i.i.d. process.

To this day, however, the literature on the exploitation of higher-order information in the RFS filtering framework remains scarce. To the best of the authors' knowledge, the extraction of *local* information on the target number of higher order than the mean remains largely unexplored, even though it can potentially lead to the design of meaningful statistical tools, for example as supporting data in sensor management decision problems.

The aim of this paper is to produce first and second-order information statistics describing the *local* target number anywhere in the state space, using the novel derivations tools introduced in [3], [4], [5] and previously applied in the context of PHD filtering in [6], [7]. This paper shall first present a novel approach for the extraction of the intensity or localised mean target number, propagated by the usual CPHD filter. Then, it shall propose the novel extraction of a higher-order statistical quantity, namely the *localised variance in the target number*, which provides a measure of the uncertainty in the estimated target number propagated by the filter.

Section II presents general concepts on stochastic population processes that shall be needed for the construction of information statistics. Section III then describes the construction of the probability generating functional (PGFl) of the target process following a CPHD Bayes update step. Section IV provides a detailed construction of the localised mean (section IV-B) and variance (section IV-C) in the target number, drawn from functional derivatives of the PGFl. Section V summarizes the key contributions of the paper. An intermediary result is provided in the appendix.

## II. Stochastic population processes: generalities

In this paper, the objects of interest are *targets* with individual states $x$ belonging to some target space $\mathbf{X} \subset \mathbb{R}^{d_x}$, typically including position and velocity variables. The multi-object filtering framework focuses of the *target population* rather than *individual targets*; because the target number *and* the target states are unknown and (possibly) time-varying, the target population is described by a *stochastic population process or point process* $\phi$, a random variable whose realizations are sets of points $\varphi = \{x_1(\omega), \ldots, x_{N(\omega)}(\omega)\}$ depicting specific multi-target configurations.

More formally, a point process $\phi$ on $\mathbf{X}$ is a measurable mapping

$$\phi : (\Omega, \mathcal{F}, \mathbf{P}) \to (\mathcal{X}, \mathbf{B}_{\mathcal{X}}) \tag{1}$$

between a given probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the measurable space $(\mathcal{X}, \mathbf{B}_{\mathcal{X}})$, where $\mathcal{X}$ is the process state space, i.e. the space of all the sets of distinct points in $\mathbf{X}$, and $\mathbf{B}_{\mathcal{X}}$ is the Borel $\sigma$-algebra on $\mathcal{X}$. As usual with random variables, $\phi$ shall be more easily described by its probability distribution on $(\mathcal{X}, \mathbf{B}_{\mathcal{X}})$ generated by $\mathbf{P}$, denoted by $\mathcal{P}$.

In the scope of this paper, delimited by the multi-object filtering framework and the novel derivation rules described in [4], [3], the probability measure $\mathcal{P}$ allows for the description of a wide range of point processes and may be *non-symmetrical* w.r.t. the targets. It will be shown, however, that the assumptions behind the CPHD filter discard non-symmetrical probability measures and are largely responsible for the tractability of the resulting filtering equations (see section III).

The PGFl of the target process $\phi$ reads as follows [4]:

$$\begin{aligned} G_\phi[w] &= \mathbb{E}\left[\prod_{x\in\phi} w(x)\right] = \int \left(\prod_{x\in\varphi} w(x)\right) \mathcal{P}_\phi(d\varphi) \\ &= \sum_{n\geqslant 0} \int \left(\prod_{i=1}^{n} w(\bar{x}_i)\right) \mathcal{P}_\phi(d\bar{x}_{1:n}). \end{aligned} \tag{2}$$

For simplicity's sake, $\prod_{i=1}^{n} w(x_i)$ may also be written as $w(x_{1:n})$, although one must keep in mind that $w(.)$ is the usual test function defined on the target space $\mathbf{X}$.

Assuming that a collection of measurements $z_{1:m}$ is newly available, the target process $\phi$ updates to $\phi^+$ following the general Bayes' rule (Theorem 6 in [4]):

$$\begin{aligned} G_{\phi^+}[w|z_{1:m}] &= \frac{\mathbb{E}\left[\left(\prod_{x\in\phi} w(x)\right) L(z_{1:m}|\phi)\right]}{\mathbb{E}[L(z_{1:m}|\phi)]} \\ &= \frac{\sum\limits_{n\geqslant 0} \int w(\bar{x}_{1:n}) L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}{\sum\limits_{n\geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}, \end{aligned} \tag{3}$$

where $L(z_{1:m}|x_{1:n})$ is the multi-measurement / multi-target likelihood describing the (current) observation process, encapsulating the detection, measurement and clutter processes.

In the general framework described in [4], [3], the likelihood $L$ may be non-symmetrical w.r.t. the measurements $z_{1:m}$ *and/or* the targets $x_{1:n}$; However, in the case of the CPHD filter as well as in many other examples of practical multi-object filters, the order of the incoming measurements $z_{1:m}$ is arbitrary and conveys no meaning. For this reason, the likelihood $L$ shall be assumed symmetrical w.r.t. the measurements. As previously mentionned, the symmetry w.r.t. to the targets will stem from the specific assumptions of the CPHD filter in section III.

## III. The CPHD Bayes update: probability generating functional

We shall now apply to the general Bayes' rule (3) the specifics of the CPHD filter in order to produce the PGFl of the updated target process $\phi^+$, which is a critical component for the computation of the localised mean and variance (see section IV). The assumptions of the CPHD update step are the following [2]:

1) The predicted target process $\phi$ is i.i.d.;
2) A detected target produces a single measurement;
3) A measurement stems from at most one target;
4) The clutter process is i.i.d.

From assumption 1 follows the explicit construction of the probability measure $\mathcal{P}$ of the predicted process $\phi$. For any set of points $x_{1:n} \in \mathbf{X}^n$:

$$\mathcal{P}(dx_{1:n}) = \rho(n) \prod_{i=1}^{n} \frac{\mu_1(dx_i)}{\int \mu_1(d\bar{x})} = \rho(n) \prod_{i=1}^{n} \tilde{\mu}_1(dx_i), \tag{4}$$

where $\rho$, $\mu_1$, $\tilde{\mu}_1$ are respectively the cardinality probability, the first moment measure and the normalized first moment measure of $\phi$[1]. By definition of an i.i.d. process [8] the average number of targets in the whole state space, denoted by $N$, follows the relation:

$$N = \sum_{n\geqslant 1} n\rho(n) = \int \mu_1(d\bar{x}). \tag{5}$$

In addition, the probability generating function (PGF) of the predicted process will be noted $G$ such that:

$$\forall y \in [0\ 1],\ G(y) = \sum_{n\geqslant 0} \rho(n) y^n. \tag{6}$$

Note that the density associated to the first moment measure, denoted by $\mu_1$ as well, is the intensity propagated by the CPHD filter [2].

Assumptions 2 to 4 shape the multi-measurement / multi-target likelihood $L$ and yield:

$$L(z_{1:m}|x_{1:n}) = \sum_{\pi\in\Pi_{m,n}} \pi_\emptyset! \rho^c(\pi_\emptyset) \prod_{(i,\emptyset)\in\pi} c(z_i) \prod_{(i,j)\in\pi} P(z_i|x_j) \prod_{(\emptyset,j)\in\pi} P(\emptyset|x_j), \tag{7}$$

where:

- $\emptyset$ denotes the empty configuration;
- $\Pi_{m,n}$ is the set of all the partitions of indexes $\{i_1, ...i_m, j_1, ..., j_n\}$ solely composed of tuples of the form $(i_a, j_b)$ (target $x_{j_b}$ is detected and produces measurement $z_{i_a}$), $(\emptyset, j_b)$ (target $x_{j_b}$ is not detected). or $(i_a, \emptyset)$ (measurement $z_{i_a}$ is clutter);
- $\pi_\emptyset = \#\{i|(i,\emptyset) \in \pi\}$ is the number of clutter measurements given by partition $\pi$;
- $P$ is the single-measurement / single-target likelihood;
- $\rho^c$ is the cardinality distribution of the clutter process;
- $c(.)$ is the individual probability distribution of a clutter measurement.

[1] In the scope of this paper, it is implicit that the neighbourhoods $dx$ defined around any point $x \in \mathbf{X}$ are chosen as members of the Borel algebra $\mathbf{B}_\mathbf{X}$. Thus, $P(dx) = Q(dx)$ is well-defined and equivalent to $\int f(x)P(dx) = \int f(x)Q(dx)$ for any test function $f$.

In addition, the PGF of the clutter process will be noted $C$ such that:

$$\forall y \in [0\ 1],\ C(y) = \sum_{n \geqslant 0} \rho^c(n) y^n. \tag{8}$$

The assumptions characterizing the CPHD filter have a critical impact on the simplification of the Bayes update equation (3), notably because (4) (resp. (7)) imply a *symmetrical* probability measure $\mathcal{P}$ (resp. likelihood $L$) w.r.t. the targets.

As an illustration, we shall now detail the expression of the normalizing term in Bayes' rule (3) w.r.t. the predicted state of the CPHD filter, i.e. w.r.t. the PGF $G$ and the first moment $\mu_1$ of the target process $\phi$. Using the specific form of the predicted probability measure (4) gives:

$$\sum_{n \geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n}) = \sum_{n \geqslant 0} \rho(n) \int L(z_{1:m}|\bar{x}_{1:n}) \prod_{i=1}^{n} \tilde{\mu}_1(d\bar{x}_i). \tag{9}$$

Let us first fix an arbitrary target number $n \in \mathbb{N}$ and consider the quantity $\int L(z_{1:m}|\bar{x}_{1:n}) \prod_{i=1}^{n} \tilde{\mu}_1(d\bar{x}_i)$. Since the likelihood is symmetrical w.r.t. the targets, the integration variables $\bar{x}_{1:n}$ play an identical role in (9) and using the specific form of the likelihood (7) yields:

$$\int L(z_{1:m}|\bar{x}_{1:n}) \prod_{i=1}^{n} \tilde{\mu}_1(d\bar{x}_i) = \sum_{\pi \in \Pi_{m,n}} \pi_\emptyset! \rho^c(\pi_\emptyset) \prod_{(i,\emptyset) \in \pi} c(z_i) \prod_{(i,j) \in \pi} P^+(z_i) \prod_{(\emptyset,j) \in \pi} P^+(\emptyset), \tag{10}$$

where:

- $P^+(z_i) = \int P(z_i|\bar{x}) \tilde{\mu}_1(d\bar{x})$ is the probability that $z_i$ is a true measurement (i.e. non clutter);
- $P^+(\emptyset) = \int P(\emptyset|\bar{x}) \tilde{\mu}_1(d\bar{x})$ is the probability that a missed detection occurs.

In other words, measurement / target pairs $(z_i, x_{j_1})$ and $(z_i, x_{j_2})$ are equivalent for integration purpose and a partition $\pi \in \Pi_{m,n}$ consists of selecting:

- a number $d$ of detections;
- a collection of $d$ true measurements in $z_1, \ldots, z_m$;
- an *arbitrary* collection of $d$ detected targets in $x_1, \ldots, x_n$.

Therefore, (10) simplifies as follows:

$$\begin{aligned} &\int L(z_{1:m}|\bar{x}_{1:n}) \prod_{i=1}^{n} \tilde{\mu}_1(d\bar{x}_i) \\ &\propto \sum_{d=0}^{\min(m,n)} \frac{n!(m-d)!}{(n-d)!} \rho^c(m-d) (P^+(\emptyset))^{n-d} \sum_{\substack{I \subseteq z_{1:m} \\ |I|=d}} \prod_{z \in I} \frac{P^+(z)}{c(z)} \\ &\propto \sum_{d=0}^{\min(m,n)} \frac{n!(m-d)!}{(n-d)!} \rho^c(m-d) (P^+(\emptyset))^{n-d} e_d(z_{1:m}), \end{aligned} \tag{11}$$

where $e_d$ is the elementary symmetric function [8] of order $d$

$$e_d(Z) = \sum_{S \subseteq Z, |Z|=d} \left( \prod_{\xi \in S} \xi \right) \tag{12}$$

applied to the set $\left\{ \frac{P^+(z)}{c(z)} | z \in z_{1:m} \right\}$ and abusively noted $e_d(z_{1:m})$.

Thus, the denominator (9) becomes:

$$\begin{aligned} &\sum_{n \geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n}) \\ &\propto \sum_{n \geqslant 0} \rho(n) \sum_{d=0}^{\min(m,n)} \frac{n!(m-d)!}{(n-d)!} \rho^c(m-d) (P^+(\emptyset))^{n-d} e_d(z_{1:m}) \\ &\propto \sum_{d=0}^{m} \left( \sum_{n \geqslant d} \rho(n) \frac{n!(P^+(\emptyset))^{n-d}}{(n-d)!} \right) (m-d)! \rho^c(m-d) e_d(z_{1:m}) \\ &\propto \sum_{d=0}^{m} G^{(d)}(P^+(\emptyset)) C^{(m-d)}(0) e_d(z_{1:m}), \end{aligned} \tag{13}$$

where the last equation is resolved using the expressions of the PGFs (6), (8). The multiplying constant in (13), found to be $\prod_{z \in z_{1:m}} c(z)$, will appear as well in the expression of the numerator of the PGFl (see section IV) and shall be omitted from now on for simplicity's sake.

## IV. Localised mean and variance in target number

Making use of the novel derivation tools introduced in [3], [4], this sections aims at retrieving the mean and variance of the local target number from the PGFl of the updated target process $\phi^+$. The localised mean target number is provided by the first moment measure and, as such, the method exposed in section IV-B produces the same result as Mahler's original expression of the updated first moment density in the CPHD filter [2]. On the other hand, the extraction of the localised variance exposed in section IV-C is, to the best of the authors' knowledge, a novel result.

### A. Construction of local information statistics

Local information statistics provide a statistical description of the number of targets - according to the target process $\phi$ - in any region $B \in \mathbf{B_X}$ of the target space. Given the target process $\phi$, one can define the counting measure

$$\phi(B) = \sum_{x \in \phi} 1_B(x) \tag{14}$$

as an integer-valued random variable which counts[2] the number of targets within such a region $B$ [9].

Just as usual random variables, $\phi(B)$ can be described by its statistical moments. The *mean target number* $\mu_1(B)$ is directly provided by the integral of the first moment measure

[2] $1_B$ is the indicator function on $B$

$\mu_1$ of the target process $\phi$ over $B$, while the *variance in target number* $var(B)$ is given by:

$$var(B) = \mu_2(B \times B) - \mu_1(B)^2, \tag{15}$$

where $\mu_2$ is the second *non factorial* moment measure of the target process $\phi$ [9].

Note that the first moment measure $\mu_1$ is propagated by the CPHD filter, but the second moment measure $\mu_2$ must be specifically determined for the variance.

### B. Propagation of the localised mean target number

Following Corollary 2 in [4], the first moment measure $\mu_1^+$ of the updated process $\phi^+$ is retrieved from the first-order functional derivative [3] of the updated PGFl (3):

$$\mu_1^+(dx) = \delta(G_{\phi^+}[w|z_{1:m}]; 1_{dx})\big|_{w=1}$$
$$= \frac{\sum_{n\geqslant 0} \int \delta(w(\bar{x}_{1:n}); 1_{dx})|_{w=1} L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}{\sum_{n\geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}. \tag{16}$$

The notation $\delta$ designs a restricted Gâteaux differential adapted to the derivation of composite functions (see [4] for a detailed description). Using Corollary 1 in [4], the numerator in (16) expands as follows:

$$\sum_{n\geqslant 0} \int \delta(w(\bar{x}_{1:n}); 1_{dx})|_{w=1} L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})$$
$$= \sum_{n\geqslant 1} \int \left( \sum_{1\leqslant j\leqslant n} \prod_{i=1}^{n} \mu_i^j(\bar{x}_i) \right) L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n}),$$

where $\mu_i^j = 1_{dx}$ if $i = j$, $\mu_i^j = 1$ otherwise. Thus:

$$\sum_{n\geqslant 0} \int \delta(w(\bar{x}_{1:n}); 1_{dx})|_{w=1} L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})$$
$$= \sum_{n\geqslant 1} \int \sum_{1\leqslant j\leqslant n} L(z_{1:m}|\hat{x}_{1:n}^j) \mathcal{P}(d\hat{x}_{1:n}^j),$$

where $\hat{x}_i^j = x$ if $i = j$, $\hat{x}_i^j = \bar{x}_i$ otherwise. Exploiting the symmetry of $L(z_{1:m}|\bar{x}_{1:n})$ and $\mathcal{P}(d\bar{x}_{1:n})$ w.r.t. to the targets (see (4) and (7)) gives:

$$\sum_{n\geqslant 0} \int \delta(w(\bar{x}_{1:n}); 1_{dx})|_{w=1} L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})$$
$$= \sum_{n\geqslant 1} n \int L(z_{1:m}|\bar{x}_{1:n-1}, x) \mathcal{P}(d\bar{x}_{1:n-1}, dx)$$
$$= \mu_1(dx) \sum_{n\geqslant 1} \frac{n\rho(n)}{N} \int L(z_{1:m}|\bar{x}_{1:n-1}, x) \prod_{i=1}^{n-1} \tilde{\mu}_1(d\bar{x}_i). \tag{17}$$

Now, considering the expression of the likelihood (7), the likelihood term in (17) can be split between the partitions where the target $x$ is not detected and those where it is detected a produces a particular measurement $z \in z_{1:m}$:

$$L(z_{1:m}|\bar{x}_{1:n-1}, x) =$$
$$P(\emptyset|x) L(z_{1:m}|\bar{x}_{1:n-1}) + \sum_{z\in z_{1:m}} P(z|x) L(z_{1:m} \setminus z|\bar{x}_{1:n-1}). \tag{18}$$

Substituting (17) and (18) into the expression of the updated first moment measure (16) finally yields:

$$\mu_1^+(dx) = \mu_1(dx) \left[ P(\emptyset|x) L_1(\emptyset) + \sum_{z\in z_{1:m}} \frac{P(z|x)}{c(z)} L_1(z) \right], \tag{19}$$

where the corrector terms $L_1(\emptyset)$ and $L_1(z)$, developed in a similar way as shown in (13), are found to be:

$$L_1(\emptyset) = \frac{\sum_{n\geqslant 1} \frac{n\rho(n)}{N} \int L(z_{1:m}|\bar{x}_{1:n-1}) \prod_{i=1}^{n-1} \tilde{\mu}_1(d\bar{x}_i)}{\sum_{n\geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}$$
$$= \frac{N^{-1} \sum_{d=0}^{m} G^{(d+1)}(P^+(\emptyset)) C^{(m-d)}(0) e_d(z_{1:m})}{\sum_{d=0}^{m} G^{(d)}(P^+(\emptyset)) C^{(m-d)}(0) e_d(z_{1:m})}, \tag{20}$$

and:

$$L_1(z) = \frac{c(z) \sum_{n\geqslant 1} \frac{n\rho(n)}{N} \int L(z_{1:m} \setminus z|\bar{x}_{1:n-1}) \prod_{i=1}^{n-1} \tilde{\mu}_1(d\bar{x}_i)}{\sum_{n\geqslant 0} \int L(z_{1:m}|\bar{x}_{1:n}) \mathcal{P}(d\bar{x}_{1:n})}$$
$$= \frac{N^{-1} \sum_{d=0}^{m-1} G^{(d+1)}(P^+(\emptyset)) C^{(m-d-1)}(0) e_d(z_{1:m} \setminus z)}{\sum_{d=0}^{m} G^{(d)}(P^+(\emptyset)) C^{(m-d)}(0) e_d(z_{1:m})}. \tag{21}$$

As expected, the expression of the updated first moment (19) is identical to Mahler's original result (equation (63) in [2]). For comparison's sake, note that $N = G^{(1)}(1)$ from the definition of the PGF (6), and that we apply the elementary symmetric functions $e_d$ to sets of the form $\left\{ \frac{P^+(z)}{c(z)} | z \in z_{1:m} \right\}$ rather than Mahler's $\left\{ \frac{NP^+(z)}{c(z)} | z \in z_{1:m} \right\}$.

The quantity $\mu_1^+(dx)$ can then be integrated in any region $B \in \mathbf{B_X}$ and provide the updated mean target number $\mu_1^+(B)$ in the said region:

$$\mu_1^+(B) = \mu_1^{\emptyset}(B) L_1(\emptyset) + \sum_{z\in z_{1:m}} \mu_1^z(B) L_1(z), \tag{22}$$

where:

$$\mu_1^{\emptyset}(B) = \int_B P(\emptyset|\bar{x}) \mu_1(d\bar{x}) \tag{23}$$

$$\mu_1^z(B) = c(z)^{-1} \int_B P(z|\bar{x}) \mu_1(d\bar{x}). \tag{24}$$

*C. Extraction of the localised variance in the target number*

The second moment measure $\mu_2^+$ can be retrieved from the second-order functional derivative [3] of the Laplace functional of the target process $\phi$ [9]:

$$\mu_2^+(dx \times dx') = \delta^2(G_{\phi^+}[e^{-w}|z_{1:m}]; 1_{dx}, 1_{dx'})\Big|_{w=0}$$
$$= \frac{\sum_{n\geqslant 0}\int \delta^2(e^{-\sum w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\Big|_{w=0} L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n})}{\sum_{n\geqslant 0}\int L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n})}. \quad (25)$$

The expression of the second-order derivative in (25) is found to be:

$$\delta^2(e^{-\sum_{i=1}^n w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\Big|_{w=0} = \sum_{1\leqslant j\leqslant n} 1_{dx\cap dx'}(\bar{x}_j) + \sum_{1\leqslant j_1,j_2\leqslant n}^{\neq} 1_{dx}(\bar{x}_{j_1})1_{dx'}(\bar{x}_{j_2}). \quad (26)$$

The proof is given in appendix. Substituting (26) in the numerator of the right-hand side of (25) gives:

$$\sum_{n\geqslant 0}\int \delta^2(e^{-\sum w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\Big|_{w=0} L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n})$$
$$= \sum_{n\geqslant 1}\int \left(\sum_{1\leqslant j\leqslant n} 1_{dx\cap dx'}(\bar{x}_j)\right) L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n})$$
$$+ \sum_{n\geqslant 2}\int \left(\sum_{1\leqslant j_1,j_2\leqslant n}^{\neq} 1_{dx}(\bar{x}_{j_1})1_{dx'}(\bar{x}_{j_2})\right) L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n}).$$

Now, exploiting the symmetry of $L(z_{1:m}|\bar{x}_{1:n})$ and $\mathcal{P}(d\bar{x}_{1:n})$ w.r.t. to the targets (see (4) and (7)) yields:

$$\sum_{n\geqslant 0}\int \delta^2(e^{-\sum w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\Big|_{w=0} L(z_{1:m}|\bar{x}_{1:n})\mathcal{P}(d\bar{x}_{1:n})$$
$$= \sum_{n\geqslant 1} n\int 1_{dx\cap dx'}(\bar{x})L(z_{1:m}|\bar{x}_{1:n-1},\bar{x})\mathcal{P}(d\bar{x}_{1:n-1}, d\bar{x})$$
$$+ \sum_{n\geqslant 2} n(n-1)\int L(z_{1:m}|\bar{x}_{1:n-2},x,x')\mathcal{P}(d\bar{x}_{1:n-2},dx,dx'). \quad (27)$$

The first term in (27) is developed exactly as shown in (17) and, once divided by the denominator (13), yields $\mu_1^+(dx\cap dx')$.

Using (4), the second term in (27) can be developed as follows:

$$\sum_{n\geqslant 2} n(n-1)\int L(z_{1:m}|\bar{x}_{1:n-2},x,x')\mathcal{P}(d\bar{x}_{1:n-2},dx,dx')$$
$$= \mu_1(dx)\mu_1(dx')\times$$
$$\sum_{n\geqslant 2}\frac{n(n-1)\rho(n)}{N^2}\int L(z_{1:m}|\bar{x}_{1:n-2},x,x')\prod_{i=1}^{n-2}\tilde{\mu}_1(d\bar{x}_i). \quad (28)$$

Then, considering the expression of the likelihood (7), the likelihood term in (28) can be split between the partitions where none of the targets $x$, $x'$ are detected, those where only one is detected and those where both are detected. That is:

$$L(z_{1:m}|\bar{x}_{1:n-2},x,x')$$
$$= P(\emptyset|x)P(\emptyset|x')L(z_{1:m}|\bar{x}_{1:n-2})$$
$$+ P(\emptyset|x)\sum_{z\in z_{1:m}} P(z|x')L(z_{1:m}\setminus z|\bar{x}_{1:n-2})$$
$$+ P(\emptyset|x')\sum_{z\in z_{1:m}} P(z|x)L(z_{1:m}\setminus z|\bar{x}_{1:n-2})$$
$$+ \sum_{z,z'\in z_{1:m}}^{\neq} P(z|x)P(z'|x')L(z_{1:m}\setminus z,z'|\bar{x}_{1:n-2}). \quad (29)$$

Substituting (29) into (28), then (27) into the expression of the updated second moment measure (25) finally yields:

$$\mu_2^+(dx\times dx')$$
$$= \mu_1^+(dx\cap dx') + P(\emptyset|x)\mu_1(dx)P(\emptyset|x')\mu_1(dx')L_2(\emptyset)$$
$$+ \left[P(\emptyset|x)\mu_1(dx)\sum_{z\in z_{1:m}}\frac{P(z|x')\mu_1(dx')}{c(z)}L_2(z)\right]$$
$$+ \left[P(\emptyset|x')\mu_1(dx')\sum_{z\in z_{1:m}}\frac{P(z|x)\mu_1(dx)}{c(z)}L_2(z)\right]$$
$$+ \left[\sum_{z,z'\in z_{1:m}}^{\neq}\frac{P(z|x)\mu_1(dx)}{c(z)}\frac{P(z'|x')\mu_1(dx')}{c(z')}L_2(z,z')\right], \quad (30)$$

where the corrector terms $L_2(\emptyset)$, $L_2(z)$, and $L_2(z,z')$, developed in a similar way as shown in (13), are found to be:

$$L_2(\emptyset) = \frac{N^{-2}\sum_{d=0}^{m} G^{(d+2)}(P^+(\emptyset))C^{(m-d)}(0)e_d(z_{1:m})}{\sum_{d=0}^{m} G^{(d)}(P^+(\emptyset))C^{(m-d)}(0)e_d(z_{1:m})} \quad (31)$$

$$L_2(z) = \frac{N^{-2}\sum_{d=0}^{m-1} G^{(d+2)}(P^+(\emptyset))C^{(m-d-1)}(0)e_d(z_{1:m}\setminus z)}{\sum_{d=0}^{m} G^{(d)}(P^+(\emptyset))C^{(m-d)}(0)e_d(z_{1:m})} \quad (32)$$

$$L_2(z,z') = \frac{N^{-2}\sum_{d=0}^{m-2} G^{(d+2)}(P^+(\emptyset))C^{(m-d-2)}(0)e_d(z_{1:m}\setminus z,z')}{\sum_{d=0}^{m} G^{(d)}(P^+(\emptyset))C^{(m-d)}(0)e_d(z_{1:m})}. \quad (33)$$

Using the definition of the variance (15), the integration of the second moment measure (30) over $B\times B$ and the first moment measure (19) over $B$ finally yields the variance $var^+(B)$:

$$\begin{aligned} &var^+(B) \\ &= \mu_1^+(B) + \mu_1^\emptyset(B)^2 \left[L_2(\emptyset) - L_1(\emptyset)^2\right] \\ &+ 2\mu_1^\emptyset(B) \sum_{z\in z_{1:m}} \mu_1^z(B) \left[L_2(z) - L_1(z)L_1(\emptyset)\right] \\ &+ \sum_{z,z'\in z_{1:m}} \mu_1^z(B)\mu_1^{z'}(B) \left[L_2^{\neq}(z,z') - L_1(z)L_1(z')\right], \end{aligned} \quad (34)$$

where $L_2^{\neq}(z,z') = L_2(z,z')$ if $z \neq z'$, zero otherwise.

Note the similarities between the expressions of the updated first moment measure (22) and the updated variance (34); most of the terms needed for the computation of the variance are indeed required for the propagation of the first moment measure $\mu_1$ in the usual CPHD filter. The additional corrector terms $L_2(\emptyset)$ and $L_2(z)$ are close to $L_1(\emptyset)$ and $L_1(z)$ in the sense that they do not involve additional elementary symmetric functions $e_d$, and could be computed in parallel with a reasonable additional cost. The computation of the corrector term $L_2(z,z')$, however, requires the computation of the new elementary symmetric functions $e_d(z,z')$ for every pair of distinct measurements $(z,z')$.

Overall, the additional computation of the localised variance alongside the usual CPHD update should increase the complexity of the filter of one order of magnitude to $O(m^3 \log m)$, where $m$ is the number of new measurements, although a more detailed study involving a practical implementation should be necessary to conclude on this point. In any case, the expression of the localised variance is significantly simplified in the restrictive case of the PHD filter; an analogous study for the PHD filter had been previously conducted and successfully tested on simulated data [7].

## V. Conclusion

In this paper, the multi-object filtering framework and the novel derivation rules proposed in [3], [4] have been successfully applied to produce high-order information statistics on the output of the Cardinalized Probability Hypothesis Density (CPHD) filter. In addition to providing the updated first moment density (or intensity) of the propagated multi-target density through a novel approach, this paper describes the construction of a novel second-order information statistic, the localised variance in the target number within any region of the state space.

Extracted at each iteration, the localised variance is a meaningful statistical tool which provides a measure of the uncertainty associated to the estimated mean target number given by the first moment density. Future developments should include a detailed study of the expression of the variance, notably in order to determine and analyse its boundaries, and an implementation of the results presented in this paper to approach sensor management decision problems with a CPHD filter.

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## Appendix

Expansion of $\delta^2(e^{-\sum_{i=1}^n w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\big|_{w=0}$ (see (26)).

*Proof:* Expanding the exponential gives:

$$\begin{aligned} &\delta^2(e^{-\sum_{i=1}^n w(\bar{x}_i)}; 1_{dx}, 1_{dx'})\Big|_{w=0} \\ &= \sum_{p\geqslant 0} \frac{(-1)^p}{p!}\, \delta^2\left(\left(\sum_{i=1}^n w(\bar{x}_i)\right)^p; 1_{dx}, 1_{dx'}\right)\Bigg|_{w=0} \\ &= \sum_{p\geqslant 0} \frac{(-1)^p}{p!} \sum_{p_1+\cdots+p_n=p} \binom{p}{p_{1:n}}\, \delta^2\left(\prod_{i=1}^n (w(\bar{x}_i))^{p_i}; 1_{dx}, 1_{dx'}\right)\Bigg|_{w=0}, \end{aligned}$$

where $\binom{p}{p_{1:n}}$ is the multinomial

$$\binom{p}{p_{1:n}} = \binom{p}{p_1,\ldots,p_n} = \frac{p!}{p_1!\ldots p_n!}. \quad (35)$$

Besides, using Corollary 1 in [4] gives:

$$\begin{aligned} &\delta^2\left(\prod_{i=1}^n (w(\bar{x}_i))^{p_i}; 1_{dx}, 1_{dx'}\right)\Bigg|_{w=0} \\ &= \sum_{p_j\geqslant 2} 2\binom{p_j}{2} 1_{dx}(\bar{x}_j) 1_{dx'}(\bar{x}_j) 0^{\sum p_i - 2} \\ &+ \sum_{\substack{p_{j_1},p_{j_2}\geqslant 1 \\ j_1\neq j_2}} \binom{p_{j_1}}{1}\binom{p_{j_2}}{1} 1_{dx}(\bar{x}_{j_1}) 1_{dx'}(\bar{x}_{j_2}) 0^{\sum p_i - 2}. \end{aligned}$$

Thus, it follows that:

$$\begin{aligned} &\sum_{p\geqslant 0} \frac{(-1)^p}{p!} \sum_{p_1+\cdots+p_n=p} \binom{p}{p_{1:n}}\, \delta^2\left(\prod_{i=1}^n (w(\bar{x}_i))^{p_i}; 1_{dx}, 1_{dx'}\right)\Bigg|_{w=0} \\ &= \frac{(-1)^2}{2!} \sum_{\substack{p_1+\cdots+p_n=2 \\ \exists j | p_j\geqslant 2}} 2\binom{2}{p_{1:n}}\binom{p_j}{2} 1_{dx\cap dx'}(\bar{x}_j) \\ &+ \frac{(-1)^2}{2!} \sum_{\substack{p_1+\cdots+p_n=2 \\ \exists j_1\neq j_2 | p_{j_1},p_{j_2}\geqslant 1}} \binom{2}{p_{1:n}}\binom{p_{j_1}}{1}\binom{p_{j_2}}{1} 1_{dx}(\bar{x}_{j_1}) 1_{dx'}(\bar{x}_{j_2}) \\ &= \frac{1}{2} \sum_{1\leqslant j\leqslant n} 2\binom{2}{2,0}\binom{2}{2} 1_{dx\cap dx'}(\bar{x}_j) \\ &+ \frac{1}{2} \sum_{1\leqslant j_1,j_2\leqslant n}^{\neq} \binom{2}{1,1}\binom{1}{1}\binom{1}{1} 1_{dx}(\bar{x}_{j_1}) 1_{dx'}(\bar{x}_{j_2}) \\ &= \sum_{1\leqslant j\leqslant n} 1_{dx\cap dx'}(\bar{x}_j) + \sum_{1\leqslant j_1,j_2\leqslant n}^{\neq} 1_{dx}(\bar{x}_{j_1}) 1_{dx'}(\bar{x}_{j_2}). \end{aligned}$$

■

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