Exercise sheet 2

Exercise 1. Write the nearly-constant velocity model for an object living in the 2-dimensional Euclidean plane $\mathbb{R}^2$ instead of the real line $\mathbb{R}$ as in Section 2.2.3. Use a state of the form

$$\big(\boldsymbol{x}_k^{(1)}, \boldsymbol{x}_k^{(2)}, \dot{\boldsymbol{x}}_k^{(1)}, \dot{\boldsymbol{x}}_k^{(2)}\big)^\intercal$$

with $\boldsymbol{x}_k^{(i)}$ and $\dot{\boldsymbol{x}}_k^{(i)}$ representing the position and velocity in the i-th direction, and assume that the duration of a time step is Δ.

In the following exercises, we will consider the constant DLM of dimension $d = 1$ with transition matrix $F = 1$, observation matrix $H = 1$ and with given transition and noise covariance U and V (these are all 1×1 matrices).

Exercise 2.

1. Write the corresponding state and observation equations and state explicitly the assumptions on the noise terms.
2. Assuming that $\boldsymbol{\theta}_{k-1}$ given $\boldsymbol{y}_{0:k-1} = y_{0:k-1}$ is distributed according to $\mathrm{N}(\cdot\,; \hat{m}_{k-1}, \hat{P}_{k-1})$, write the simplified equations of the Kalman filter for the special case under consideration. Write the updated mean $\hat{m}_k$ as a weighted average between $\hat{m}_{k-1}$ and y_k.
3. Give the predictive distribution of $\boldsymbol{y}_k$ given $\boldsymbol{y}_{0:k-1} = y_{0:k-1}$. Comment on the connection between the obtained variance and the covariance of the innovation in the Kalman filter.

Exercise 3.

1. Express the updated covariance $\hat{P}_k$ at time step k as a function of $\hat{P}_{k-1}$, U and V. Similarly, express the Kalman gain K_k at time step k as a function of K_{k-1}, U and V.
2. Show that the Kalman gain verifies

$$0 < \frac{K_k - K_{k-1}}{K_{k-1} - K_{k-2}} < 1$$

and conclude that the sequence $(K_k)_{k \geq 0}$ is monotonic and has a unique limit K.

Exercise 4. Consider the situation where the observation y_k is missing. Find the distribution of $\boldsymbol{\theta}_{k+1}$ given $\boldsymbol{y}_{0:k-1} = y_{0:k-1}$, and hence or otherwise find the Kalman gain for the update at time step $k + 1$.

Exercise 5. It is now assumed that the sequence of random variables $(\boldsymbol{y}_k)_{k \geq 0}$ represents the sales of a certain product. Using the state and observation equations (from Exercise 2.1) or otherwise, find the distribution of the aggregate future sales

$$A_{k,\delta} = \sum_{i=1}^{\delta} \boldsymbol{y}_{k+i}$$

given $\boldsymbol{y}_{0:k} = y_{0:k}$ for some lag $\delta > 0$.