Exercise sheet 4

Exercise 1. Consider the first-order polynomial trend model with state $\boldsymbol{\theta}_k$ and transition noise covariance matrix U_k at time step k and assume that the posterior distribution of $\boldsymbol{\theta}_{k-1}$ given $\boldsymbol{y}_{0:k-1} = y_{0:k-1}$ is normal with mean $\hat{m}_{k-1}$ and variance $\hat{P}_{k-1}$. We want to use an intervention at time step k to obtain a predicted distribution for $\boldsymbol{\theta}_k$ at time step k of the form

$$\mathrm{N}(\cdot;\, \hat{m}_{k-1}, P_k + \tilde{U}_k)$$

with $P_k = \hat{P}_{k-1} + U_k$ and with $\tilde{U}_k$ some extra variance.

i) Define the distribution of the noise $\boldsymbol{u}_k$ at time step k that encompasses all the uncertainty in the prediction.

ii) Show that an alternative approach is to modify the state equation at time step k to

$$\boldsymbol{\theta}_k = \breve{F}_k \boldsymbol{\theta}_{k-1} + c_k + \breve{\boldsymbol{u}}_k, \qquad \breve{\boldsymbol{u}}_k \sim \mathrm{N}(\cdot\,; 0, \breve{U}_k).$$

with

$$\breve{F}_k = \sqrt{1 + \frac{\tilde{U}_k}{P_k}}, \qquad c_k = (1 - \breve{F}_k)\hat{m}_{k-1} \qquad \text{and} \qquad \breve{U}_k = \Big(1 + \frac{\tilde{U}_k}{P_k}\Big) U_k.$$

Exercise 2. There were two ways of implementing the considered intervention in Exercise 1. Can you think of a situation where simply adding an independent error as in 1.i is not going to work, but we can still use the change of model considered in 1.ii? Briefly discuss in the context of a first-order polynomial trend model.