Exercise sheet 5

Exercise 1. *Consider the constant time-series DLM of dimension $d = 1$ defined as*

$$\begin{aligned}\boldsymbol{\theta}_k &= F\boldsymbol{\theta}_{k-1} + \boldsymbol{u}_k, \qquad \boldsymbol{u}_k \sim \mathrm{N}(\cdot\,; 0, U)\\ \boldsymbol{y}_k &= \boldsymbol{\theta}_k + \boldsymbol{v}_k, \qquad \boldsymbol{v}_k \sim \mathrm{N}(\cdot\,; 0, V).\end{aligned}$$

i) *Show that $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$ is a stationary Gaussian process iff $|F| < 1$, and determine its autocorrelation ρ_δ, for any $\delta \geq 0$, when $|F| < 1$. If $|F| \geq 1$, show that the derived time series $(\boldsymbol{y}_k - F\boldsymbol{y}_{k-1})_{k\in\mathbb{Z}}$ is stationary and find the corresponding autocorrelation ρ'_δ.*

ii) *Express $\boldsymbol{y}_{k+\delta}$ as a function of $\boldsymbol{\theta}_k$ and the noise terms and, assuming that $|F| < 1$, deduce that*

$$\lim_{\delta\to\infty} \mathbb{E}(\boldsymbol{y}_{k+\delta} \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) = 0 \qquad \textit{and} \qquad \lim_{\delta\to\infty} \mathrm{var}(\boldsymbol{y}_{k+\delta} \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) = \frac{1}{1-F^2}U + V.$$

Exercise 2. *Show that the following results hold for any time series $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$.*

i) $\boldsymbol{y}_k - \boldsymbol{y}_{k-1} = (1 - B)\boldsymbol{y}_k$

ii) $\boldsymbol{y}_k - \alpha\boldsymbol{y}_{k-1} = (1 - \alpha B)\boldsymbol{y}_k$ *for any $\alpha \in \mathbb{R}$*

iii) *For some $n > 0$,*

$$\sum_{i=0}^{n} \phi_i \boldsymbol{y}_{k-i} = \phi(B)\boldsymbol{y}_k$$

where $\phi(B) = \sum_{i=0}^{n} \phi_i B^i$ and $\phi_0, \ldots, \phi_n$ are arbitrary coefficients.

iv) $(1 - \alpha B)\boldsymbol{y}_k = a_k$ *for any $k \in \mathbb{Z}$ is equivalent to*

$$\boldsymbol{y}_k = (1 - \alpha B)^{-1} a_k = \sum_{i\geq 0} \alpha^i a_{k-i}$$

under the assumption that $|\alpha| < 1$ and $|a_k| < \infty$ for any $k \in \mathbb{Z}$. [Hint: $B^{-\delta}\boldsymbol{y}_k = \boldsymbol{y}_{k+\delta}$ for any $\delta \geq 0$]

Exercise 3. *Consider the constant time-series DLM of dimension $d = 1$ defined as*

$$\begin{aligned}\boldsymbol{\theta}_k &= \boldsymbol{\theta}_{k-1} + \boldsymbol{u}_k, \qquad \boldsymbol{u}_k \sim \mathrm{N}(\cdot\,; 0, U)\\ \boldsymbol{y}_k &= \boldsymbol{\theta}_k + \boldsymbol{v}_k, \qquad \boldsymbol{v}_k \sim \mathrm{N}(\cdot\,; 0, V).\end{aligned}$$

and the time series $(\boldsymbol{y}'_k)_{k\in\mathbb{Z}}$ defined by

$$\boldsymbol{y}'_k = \boldsymbol{y}'_{k-1} + \boldsymbol{\epsilon}_k - \alpha\boldsymbol{\epsilon}_{k-1},$$

where $\alpha \in \mathbb{R}$ and $(\boldsymbol{\epsilon}_k)_{k\in\mathbb{Z}}$ is an independent process with identical distribution $\mathrm{N}(\cdot\,; 0, \sigma^2)$. Show that the auto-covariance of the derived processes $(\boldsymbol{y}_k - \boldsymbol{y}_{k-1})_{k\in\mathbb{Z}}$ and $(\boldsymbol{y}'_k - \boldsymbol{y}'_{k-1})_{k\in\mathbb{Z}}$ vanish at the same lag $\delta > 0$. By identifying these two auto-covariances, find U and V as functions of α and σ so that these two derived processes have the same distribution.

Exercise 4. *Consider the* $\mathrm{ARMA}(1, 1)$ *process defined by the equation*

$$\boldsymbol{y}_k = \phi_1\boldsymbol{y}_{k-1} + \psi_1\boldsymbol{\epsilon}_{k-1} + \boldsymbol{\epsilon}_k$$

with $\phi_1, \psi_1 \in [0, 1]$. Write the process $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$ as an infinite-order moving average process.

Exercise 5. *Show that the two* MA(1) *processes* $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$ *and* $(\boldsymbol{y}'_k)_{k\in\mathbb{Z}}$ *defined by*

$$\boldsymbol{y}_k = \boldsymbol{\epsilon}_k + \frac{1}{2}\boldsymbol{\epsilon}_{k-1}, \qquad \boldsymbol{\epsilon}_k \sim \mathrm{N}(\cdot\,; 0, 4), \ \textit{for any } k \in \mathbb{Z}$$

and

$$\boldsymbol{y}'_k = \boldsymbol{\epsilon}'_k + 2\boldsymbol{\epsilon}'_{k-1}, \qquad \boldsymbol{\epsilon}'_k \sim \mathrm{N}(\cdot\,; 0, 1), \ \textit{for any } k \in \mathbb{Z}$$

have identical distributions.