Exercise sheet 6

Algorithm 1 Importance sampling

for $i = 1, \dots, N$ **do**
  $\boldsymbol{\theta}^{(i)} \sim s(\cdot)$
end for
Output:

$$\hat{\boldsymbol{I}}(\varphi) = \frac{1}{N} \sum_{i=1}^{N} \varphi(\boldsymbol{\theta}^{(i)}) w(\boldsymbol{\theta}^{(i)})$$

with $w(\theta) = \pi(\theta)/s(\theta)$

Exercise 1. Consider the importance sampling algorithm as detailed in Algorithm 1, with target distribution $\pi(\cdot)$ and proposal distribution $s(\cdot)$.

i) How would you modify Algorithm 1 to adapt it to the case where the target distribution $\pi(\cdot)$ has an unknown normalising constant?

ii) Give an unbiased estimate of the normalising constant and verify its unbiasedness.

iii) Is the obtained estimator of $I(\varphi) = \int \varphi(\theta)\pi(\theta)\,\mathrm{d}\theta$ unbiased? Justify your answer.

Exercise 2. Consider the following smoothing distribution:

$$\pi_n(\theta_{0:n}) = \frac{\gamma_n(\theta_{0:n})}{Z_n}$$

with

$$\gamma_n(\theta_{0:n}) = p_0(\theta_0)\ell_0(y_0 \mid \theta_0) \prod_{k=1}^{n} q_k(\theta_k \mid \theta_{k-1})\ell_k(y_k \mid \theta_k)$$

and with Z_n the corresponding normalising constant. It is assumed that one can sample from $p_0(\cdot)$ and $q_k(\cdot \mid \theta)$ for any $\theta \in \Theta$.

i) Why is Z_n difficult to compute in general?

ii) Which proposal distribution $s_n(\cdot)$ would you use?

iii) Propose a way to get a sample $\boldsymbol{\theta}_{0:n}$ from $s_n(\cdot)$.

iv) In the context of self-normalised importance sampling, what would be weight $\tilde{w}(\boldsymbol{\theta}_{0:n}) = \gamma_n(\boldsymbol{\theta}_{0:n})/s_n(\boldsymbol{\theta}_{0:n})$ of this sample?

v) Write the sequential importance sampling algorithm corresponding to your choice of proposal distribution.