Exercise sheet 3

Exercise 1. Consider a DLM with transition and observation matrices F_k and H_k, with transition and observation covariance matrices U_k and V_k and with $(\boldsymbol{y}_k)_{k\geq 0}$ the associated observations. The sequence of states $(\boldsymbol{\theta}_k)_{k\geq 0}$ is initialised with $\boldsymbol{\theta}_0 \sim \mathrm{N}(\cdot\,; m_0, P_0)$. We are interested in the alternative sequence of states $(\boldsymbol{\theta}'_k)_{k\geq 0}$ with $\boldsymbol{\theta}'_k = S_k\boldsymbol{\theta}_k$ for any $k \geq 0$ where S_k is a non-singular matrix. Write the state and observation equations for the DLM with states $(\boldsymbol{\theta}'_k)_{k\geq 0}$ and observations $(\boldsymbol{y}_k)_{k\geq 0}$ and express the posterior mean and variance of $\boldsymbol{\theta}'_k$ as a function of $\hat{m}_k = \mathbb{E}(\boldsymbol{\theta}_k \mid \boldsymbol{y}_{0:k} = y_{0:k})$ and $\hat{P}_k = \mathrm{var}(\boldsymbol{\theta}_k \mid \boldsymbol{y}_{0:k} = y_{0:k})$.

For the following exercises, time-series DLM are considered.

Exercise 2. Using the state and observation equations of each of the following models, write the distribution of $\boldsymbol{y}_{k+\delta}$ given $\boldsymbol{y}_{0:k} = y_{0:k}$:

1. Second-order polynomial trend model with

$$\boldsymbol{u}_k = \begin{pmatrix} \boldsymbol{\epsilon}_{k,1} + \boldsymbol{\epsilon}_{k,2} \\ \boldsymbol{\epsilon}_{k,2} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\epsilon}_k \sim \mathrm{N}\left(\cdot\,; 0, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right).$$

   for some $\sigma_1, \sigma_2 > 0$, where the sequence of random variables $(\boldsymbol{\epsilon}_k)_{k\geq 1}$ is i.i.d..

2. The superposition of a first-order polynomial trend model with a first-harmonic Fourier-form seasonal model, where the transition noise $\boldsymbol{u}_k$ verifying $\boldsymbol{u}_k \sim \mathrm{N}(\cdot\,; 0, U)$ with

$$U = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma'^2 & 0 \\ 0 & 0 & \sigma'^2 \end{pmatrix}$$

   for some $\sigma, \sigma' > 0$.

Exercise 3. You are going to launch a new product whose sales will eventually taper off. You believe that a d-dimensional model with

$$H = e_d = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \text{and} \qquad F = J_d(\lambda) = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

might be appropriate to model the monthly sales with $\lambda \neq 0$ some fixed scalar. It can be shown that

$$F^\delta = \begin{pmatrix} a_1 & a_2 & \dots & a_d \\ 0 & a_1 & \dots & a_{d-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_1 \end{pmatrix} \qquad \text{with} \qquad a_{j+1} = \begin{cases} \binom{\delta}{j}\lambda^{\delta-j} & \text{if } \delta \geq j \\ 0 & \text{otherwise.} \end{cases}$$

1. Verify by induction the result regarding the expression of F^δ.
2. Assuming that $d = 3$ and that the forecast function is of the form $g_0(\delta) = A\delta^2\exp(-\rho\delta)$, find appropriate values for λ and for the vector m_0 given that highest sales are expected four months on from the launch and to total $30,000$. Is this value of λ consistent with the fact that the sales will eventually taper off?