Exercise sheet 4
Solutions

Exercise 1. i) The noise $\boldsymbol{u}_k$ has distribution $\mathrm{N}(\cdot\,;0, U_k + \tilde{U}_k)$

ii) Using the result from the lecture notes in the case of a one-dimensional state, we have

$$M_k = \frac{\sqrt{\breve{P}_k}}{\sqrt{P_k}} = \frac{\sqrt{P_k + \tilde{U}_k}}{\sqrt{P_k}} = \sqrt{1 + \tilde{U}_k/P_k}$$

so that

$$\begin{aligned}\breve{F}_k &= M_k F_k = M_k = \sqrt{1 + \tilde{U}_k/P_k}\\ c_k &= \breve{m}_k - M_k F_k \hat{m}_{k-1} = (1 - \breve{F}_k)\hat{m}_{k-1}\\ \breve{U}_k &= M_k U_k M_k^{\intercal} = (1 + \tilde{U}_k/P_k)U_k,\end{aligned}$$

as required.

The intervention which consists of adding an independent noise term (resulting in the total covariance of the transition noise being $U_k + \tilde{U}_k$) cannot be used if the desired predicted variance is "smaller" than the original predicted variance P_k. Indeed, $P_k + \tilde{U}_k < P_k$ cannot hold since $\tilde{U}_k$ is a variance and is therefore positive definite. In this sense, the change-of-moments intervention is more general. In the context of a first-order polynomial trend model, this situation corresponds to $M_k < 1$ and, thus, $\breve{F}_k < 1$, c_k is of the same sign as $\hat{m}_{k-1}$ and $\breve{U}_k < U_k$.

Exercise 2. i) The posterior distribution is characterised by

$$\begin{aligned}p_{\boldsymbol{\theta}_{1:K}|\boldsymbol{y}_{1:n}}(\theta_{1:K} \mid y_{1:n}) &\propto p_{\boldsymbol{y}_{1:n}|\boldsymbol{\theta}_{1:K}}(y_{1:n} \mid \theta_{1:K}) p_{\boldsymbol{\theta}_{1:K}}(\theta_{1:K})\\ &\propto \prod_{k=1}^{K} \theta_k^{m_k} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}\\ &\propto \prod_{k=1}^{K} \theta_k^{m_k + \alpha_k - 1}\end{aligned}$$

ii) The marginal likelihood is

$$\begin{aligned}p_{\boldsymbol{y}_{1:n}}(y_{1:n}) &= \int p_{\boldsymbol{y}_{1:n}|\boldsymbol{\theta}_{1:K}}(y_{1:n} \mid \theta_{1:K}) p_{\boldsymbol{\theta}_{1:K}}(\theta_{1:K}) \,\mathrm{d}\theta_1 \dots \mathrm{d}\theta_K\\ &= \frac{\Gamma\big(\sum_{i=1}^K \alpha_i\big)}{\prod_{k=1}^K \Gamma(\alpha_k)} \int \prod_{k=1}^{K} \theta_k^{m_k + \alpha_k - 1} \,\mathrm{d}\theta_1 \dots \mathrm{d}\theta_K = \frac{\Gamma\big(\sum_{i=1}^K \alpha_i\big)}{\Gamma\big(n + \sum_{k=1}^K \alpha_k\big)} \prod_{k=1}^{K} \frac{\Gamma(\alpha_k + m_k)}{\Gamma(\alpha_k)}\end{aligned}$$

iii) The marginal likelihood corresponding to a uniform prior is

$$p'_{\boldsymbol{y}_{1:n}}(y_{1:n}) = \frac{\Gamma(K)}{\Gamma(n+K)} \prod_{k=1}^{K} \frac{\Gamma(m_k + 1)}{\Gamma(1)}$$

so that the corresponding Bayes factor over all time steps is

$$B_{n,n} = \frac{p_{\boldsymbol{y}_{1:n}}(y_{1:n})}{p'_{\boldsymbol{y}_{1:n}}(y_{1:n})} = \frac{\Gamma\big(\sum_{i=1}^K \alpha_i\big)\Gamma(n+K)}{\Gamma\big(n + \sum_{k=1}^K \alpha_k\big)\Gamma(K)} \prod_{k=1}^{K} \frac{\Gamma(\alpha_k + m_k)\Gamma(1)}{\Gamma(\alpha_k)\Gamma(m_k + 1)}.$$