Exercise sheet 5
Solutions

Exercise 1. i) Using the state equation, we find that

$$\boldsymbol{\theta}_k = F\boldsymbol{\theta}_{k-1} + \boldsymbol{u}_k = F(F\boldsymbol{\theta}_{k-2} + \boldsymbol{u}_{k-1}) + \boldsymbol{u}_k = F^2\boldsymbol{\theta}_{k-2} + F\boldsymbol{u}_{k-1} + \boldsymbol{u}_k = [\dots] = \sum_{i\geq 0} F^i \boldsymbol{u}_{k-i}$$

so that

$$\mathbb{E}(\boldsymbol{\theta}_k) = 0 \qquad \text{and} \qquad \mathrm{var}(\boldsymbol{\theta}_k) = U\sum_{i\geq 0} F^{2i} = \frac{U}{1-F^2}$$

are constant and

$$\gamma_\delta \doteq \mathrm{cov}(\boldsymbol{\theta}_{k-\delta}, \boldsymbol{\theta}_k) = \sum_{i\geq\delta} \mathbb{E}([F^i\boldsymbol{u}_{k-\delta-i}][F^{i+\delta}\boldsymbol{u}_{k-\delta-i}]) = \frac{F^\delta U}{1-F^2}$$

does not depend on k. It follows easily that $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$ has the same properties, that is

$$\mathbb{E}(\boldsymbol{y}_k) = 0 \qquad \text{and} \qquad \mathrm{var}(\boldsymbol{y}_k) = \frac{U}{1-F^2} + V$$

are constant and $\mathrm{cov}(\boldsymbol{y}_{k-\delta}, \boldsymbol{y}_k) = \gamma_\delta$ does not depend on k, so that both time series are weakly stationary. Since $\boldsymbol{\theta}_k$ and $\boldsymbol{y}_k$ are linear combinations of Gaussian random variable, they are also Gaussian and we can conclude that the corresponding time series are also stationary (in the "strong" sense). The autocorrelation of $(\boldsymbol{y}_k)_{k\in\mathbb{Z}}$ is

$$\rho_\delta = \frac{\mathrm{cov}(\boldsymbol{y}_{k-\delta}, \boldsymbol{y}_k)}{\sqrt{\mathrm{var}(\boldsymbol{y}_{k-\delta})}\sqrt{\mathrm{var}(\boldsymbol{y}_k)}} = \Big(\frac{F^\delta U}{1-F^2}\Big)\Big(\frac{U}{1-F^2} + V\Big)^{-1} = \frac{F^\delta U}{U + (1-F^2)V}.$$

If $|F| \geq 1$, then $\mathrm{var}(\boldsymbol{\theta}_k) = F^2\,\mathrm{var}(\boldsymbol{\theta}_{k-1}) + U > \mathrm{var}(\boldsymbol{\theta}_{k-1})$ so that $\mathrm{var}(\boldsymbol{y}_k)$ increases when k increases and the corresponding time series cannot be stationary. However, if we consider the derived time series $(\boldsymbol{y}_k - F\boldsymbol{y}_{k-1})_{k\in\mathbb{Z}}$, we find that $\boldsymbol{y}'_k \doteq \boldsymbol{y}_k - F\boldsymbol{y}_{k-1} = \boldsymbol{u}_k + \boldsymbol{v}_k - F\boldsymbol{v}_{k-1}$ so that

$$\mathbb{E}(\boldsymbol{y}'_k) = 0 \qquad \text{and} \qquad \mathrm{var}(\boldsymbol{y}'_k) = U + (1+F^2)V$$

and

$$\mathrm{cov}(\boldsymbol{y}'_{k-\delta}, \boldsymbol{y}'_k) = \begin{cases} -FV & \text{if } \delta = 1 \\ 0 & \text{if } \delta > 1. \end{cases}$$

It follows that $(\boldsymbol{y}'_k)_{k\in\mathbb{Z}}$ is weakly stationary, but since it is Gaussian we can conclude that it is also stationary. The associated autocorrelation is

$$\rho'_\delta = \begin{cases} -FV/(U + (1+F^2)V) & \text{if } \delta = 1 \\ 0 & \text{if } \delta > 1. \end{cases}$$

ii) The expression of $\boldsymbol{y}_{k+\delta}$ as a function of $\boldsymbol{\theta}_k$ and the noise terms is

$$\boldsymbol{y}_{k+\delta} = F^\delta\boldsymbol{\theta}_k + \sum_{i=0}^{\delta-1} F^i\boldsymbol{u}_{k+\delta-i} + \boldsymbol{v}_{k+\delta}$$

from which we can conclude that

$$\lim_{\delta\to\infty} \mathbb{E}(\boldsymbol{y}_{k+\delta} \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) = \lim_{\delta\to\infty} F^{\delta}\mathbb{E}(\boldsymbol{\theta}_k \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) = 0$$

and

$$\begin{aligned}\lim_{\delta\to\infty} \mathrm{var}(\boldsymbol{y}_{k+\delta} \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) &= \lim_{\delta\to\infty} \Big(F^{2\delta}\,\mathrm{var}(\boldsymbol{\theta}_k \,|\, \boldsymbol{y}_{0:k} = y_{0:k}) + \sum_{i=0}^{\delta-1} F^{2i}\,\mathrm{var}(\boldsymbol{u}_{k-i})\Big) + V \\ &= \frac{U}{1-F^2} + V\end{aligned}$$

Exercise 2. i) Using the definition of the backshift operator, i.e. $B\boldsymbol{y}_k = \boldsymbol{y}_{k-1}$, it simply follows that

$$\boldsymbol{y}_k - \alpha\boldsymbol{y}_{k-1} = \boldsymbol{y}_k - \alpha B\boldsymbol{y}_k = (1-\alpha B)\boldsymbol{y}_k$$

ii) It also holds that

$$\sum_{i=0}^{n} \phi_i \boldsymbol{y}_{k-i} = \sum_{i=0}^{n} \phi_i B^i \boldsymbol{y}_k = \Big(\sum_{i=0}^{n} \phi_i B^i\Big)\boldsymbol{y}_k = \phi(B)\boldsymbol{y}_k$$

iii) The inverse B^{-1} of the operator B is characterised by $B^{-1}B\boldsymbol{y}_k = BB^{-1}\boldsymbol{y}_k = \boldsymbol{y}_k$. It indeed follows that the relation $B\boldsymbol{y}_k = \boldsymbol{y}_{k-1}$ when multiplied on both sides by B^{-1} gives $B^{-1}\boldsymbol{y}_{k-1} = \boldsymbol{y}_k$, as suggested in the hint. It then holds that

$$(1-\alpha B)\boldsymbol{y}_k = a_k \iff \boldsymbol{y}_k = a_k + \alpha\boldsymbol{y}_{k-1} = a_k + \alpha a_{k-1} + \alpha^2\boldsymbol{y}_{k-2} = \sum_{i\geq 0} \alpha^i a_{k-i}.$$

This sum is finite iff $|\alpha| < 1$ and all the a_k are finite, in which case it holds that

$$\boldsymbol{y}_k = \sum_{i\geq 0} \alpha^i a_{k-i} = \Big(\sum_{i\geq 0} (\alpha B)^i\Big) a_k = (1-\alpha B)^{-1} a_k$$

Exercise 3. Using the result of Exercise 1, we find that the auto-covariance of the derived process $(\boldsymbol{y}_k - \boldsymbol{y}_{k-1})_{k\in\mathbb{Z}}$ is equal to $U + 2V$ for a lag of 0, to $-V$ for a lag of 1 and to 0 for larger lags. The auto-covariance of the derived process $(\boldsymbol{y}'_k - \boldsymbol{y}'_{k-1})_{k\in\mathbb{Z}}$ also vanishes for lags strictly greater than 1 and is equal to $(1+\alpha^2)\sigma^2$ for a lag of 0 and to $-\alpha\sigma^2$ for a lag of 1. By matching these auto-covariances, we find that $U = \sigma^2(1-\alpha)^2$ and $V = \alpha\sigma^2$.

Exercise 4. Expressing this ARMA$(1,1)$ process as a MA process can be done efficiently by using the backshift operator, that is by expressing the equation of the process as $(1-\phi_1 B)\boldsymbol{y}_k = (1+\psi_1 B)\boldsymbol{\epsilon}_k$ from which it follows that

$$\begin{aligned}\boldsymbol{y}_k &= (1-\phi_1 B)^{-1}(1+\psi_1 B)\boldsymbol{\epsilon}_k \\ &= \Big(\sum_{i\geq 0} \phi_1^i B^i\Big)(1+\psi_1 B)\boldsymbol{\epsilon}_k \\ &= \sum_{i\geq 0} \phi_1^i \boldsymbol{\epsilon}_{k-i} + \sum_{i\geq 0} \phi_1^i \psi_1 \boldsymbol{\epsilon}_{k-i-1} \\ &= \boldsymbol{\epsilon}_k + \sum_{i\geq 1} \phi_1^i \boldsymbol{\epsilon}_{k-i} + \sum_{j\geq 1} \phi_1^{j-1}\psi_1 \boldsymbol{\epsilon}_{k-j} \\ &= \boldsymbol{\epsilon}_k + \sum_{i\geq 1} \phi_1^{i-1}(\phi_1+\psi_1)\boldsymbol{\epsilon}_{k-i}\end{aligned}$$

so that $\boldsymbol{y}_k = \sum_{i\geq 0} \phi'_i \boldsymbol{\epsilon}_{k-i}$ with $\phi'_0 = 1$ and $\phi'_i = \phi_1^{i-1}(\phi_1+\psi_1)$ for any $i \geq 1$.

Exercise 5. It is easily seen that each process has zero mean, variance 5 and δ-lag auto-covariance equals to 2 for $\delta = 1$ and to 0 for $\delta > 1$. Since each process is normal, it follows that they have identical joint distributions.